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A note on an algebraic instability of inviscid parallel shear flows

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It is shown that all parallel inviscid shear flows of constant density are unstable to a wide class of initial infinitesimal three-dimensional disturbances in the sense that, according to linear theory, the kinetic energy of the disturbance will grow at least as fast as linearly in time. This can occur even when the disturbance velocities are bounded, because the streamwise length of the disturbed region grows linearly with time. This finding may have implications for the observed tendency of turbulent shear flows to develop a longitudinal streaky structure.

1. Introduction

The question of under what conditions a parallel shear flow will become unstable has been one of the central problems in hydrodynamic research since the finding by Lord Rayleigh (1880) that a necessary condition for a parallel inviscid shear flow to have wavelike disturbances which grow exponentially with time is that the velocity profile possesses an inflexion point. Fjørtoft (1950) later showed that the inflexion point must correspond to a maximum in the shear. An important general result was also obtained by Howard (1961), who proved that the complex phase velocity of the wavelike disturbance must lie within a semicircle having a diameter equal to the difference between the largest and smallest velocity in the parallel flow. Arnol'd (1965) demonstrated that Rayleigh's inflexion point criterion also holds for finite disturbances. Such general results afford a good overall understanding of the qualitative features of the stability of a disturbance in form of an infinite wave train. For a more detailed quantitative description one needs of course the actual eigenvalues, which may be found analytically for simple velocity profiles, such as those formed by straight-line segments, or by numerical methods. For a comprehensive discussion of such aspects we refer to the review article by Drazin & Howard (1966).

Less well understood, however, is the evolution in time and space of an arbitrary three-dimensional disturbance introduced in the flow at some initial instant. A discussion of the initial-value problem was given as early as in Orr's paper (1907), in which he pointed out the necessity of including a continuous as well as a discrete spectrum in the specification of the initial disturbance, a fact that was often overlooked in subsequent treatments of this problem. Eliassen, Høiland & Riis (1953), however, took the continuous spectrum into account and gave a detailed analysis of the initial-value problem for an inviscid Couette flow. They were able to show that the disturbance velocity component in the direction normal to the flow vanishes at least as fast as t^{-2}

for $t \to \infty$. Case (1960) also independently considered the same problem. Recently, Chimonas (1979) has shown that for a shear flow with stable density stratification, disturbances growing algebraically in time may arise. All of these investigations were restricted to two-dimensional disturbances. An analysis of three-dimensional initial disturbances in a boundary-layer type inviscid parallel shear flow has recently been carried out by Gustavsson (1978). The question of the asymptotic behaviour of the disturbances at large times was addressed by Wilke (1967), who showed that for a special class of initial disturbances that are independent of the streamwise direction, the perturbation kinetic energy may reach values up to twice the kinetic energy of the unperturbed flow. Such initial disturbances of infinitesimal amplitude were also considered by Ellingsen & Palm (1975), who demonstrated that a streamwise perturbation velocity increasing linearly with time will result. They also pointed out that for finite disturbances, large gradients in u, i.e., large vorticity concentrations, would be created, thus possibly giving rise to strongly unstable flows.

The purpose of this note is to show that a wide class of localized initial three-dimensional disturbances give rise to a perturbation kinetic energy growing linearly with time for *any* inviscid shear flow. The explanation for this behaviour is that the streamwise size of the disturbed region will increase linearly in time, whereas the streamwise disturbance velocity u does not decrease as $t \to \infty$.

2. Analysis

Consider infinitesimal disturbances with velocity components u, v, w on a parallel inviscid shear flow with velocity $\mathbf{U} = \mathbf{i}U(y)$. They are governed by

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) u + U'v = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{1}$$

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) v &= -\frac{1}{\rho}\frac{\partial p}{\partial y}, \\ \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) w &= -\frac{1}{\rho}\frac{\partial p}{\partial z}, \end{aligned}\right\} \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{4}$$

The pressure p may be eliminated by the use of

$$\nabla^2 p = -2\rho U' \frac{\partial v}{\partial x}, \tag{5}$$

obtained by taking the divergence of the vector momentum equation (1)–(3). Then the following single equation for v may be derived:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2 v - U''\frac{\partial v}{\partial x} = 0. \tag{6}$$

This is to be solved subject to the initial conditions

$$u, v, w = u_0, v_0, w_0 \quad \text{at} \quad t = 0 \tag{7}$$

and the boundary conditions that v is zero at the boundaries of the flow, at $y = 0$ and $y = d$, say. We shall assume that the initial disturbance is localized so that it vanishes outside some finite radius from the origin.

By the integration of (1) over x we find

$$\frac{\partial \bar{u}}{\partial t} = -\bar{v}U'(y), \tag{8}$$

where

$$\bar{u} = \int_{-\infty}^{\infty} u\,dx \tag{9}$$

and

$$\bar{v} = \int_{-\infty}^{\infty} v\,dx. \tag{10}$$

Here it has been assumed that the disturbance is sufficiently localized such that the integrals (9), (10) exist. By integration of (2) one finds

$$\frac{\partial \bar{v}}{\partial t} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y},$$

where

$$\bar{p} = \int_{-\infty}^{\infty} p\,dx. \tag{11}$$

From the solution of (5) it follows that p may be written as a derivative with respect to x of a finite integral, which vanishes for large x. Hence $\bar{p} = 0$, and from (11) it thus follows that $\bar{v}$ is independent of time. Integration of (8) then gives that

$$\bar{u} = \bar{u}_0 - t\bar{v}_0 U', \tag{12}$$

i.e., for $\bar{v}_0$ and dU/dy different from zero the integrated streamwise momentum increases linearly with time. This does not imply that u itself necessarily increases, since the disturbance could – and in fact does – spread streamwise as time goes on. Thus, although the analysis for the integrated quantities is mathematically the same in the linear limit as the analysis for x-independent disturbances (Stuart 1965; Wilke 1967; Ellingsen & Palm 1975), the time over which linear theory remains valid should be much longer in the present case than in theirs (see discussion below). As will be shown subsequently, the streamwise spreading of the disturbance allows its total kinetic energy to grow with time, at least as fast as t.

In Schwartz's inequality

$$\left|\int_{-\infty}^{\infty} f(x)\,g(x)\,dx\right| \leqslant \left\{\int_{-\infty}^{\infty} |f(x)|^2 dx \int_{-\infty}^{\infty} |g(x)|^2 dx\right\}^{\frac{1}{2}}, \tag{13}$$

where f and g are any measurable and integrable functions (see Titschmarsh 1939, p. 381), we set $f = u$ and

$$g = \begin{cases} 1 & \text{for} \quad (U_{\min} - \Delta)\,t < x < (U_{\max} + \Delta)\,t, \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

where $U_{\min}$ and $U_{\max}$ are the minimum and maximum velocities, respectively, in the parallel shear flow, and where $\Delta > 0$. An analysis of the initial-value problem (see appendix) shows that the propagation speed of disturbances lies between $U_{\min}$ and $U_{\max}$. As time increases, a disturbance of finite initial streamwise extent will therefore become more and more confined within the streamwise region of non-zero g. This also includes the streamwise tails of the disturbance, which will decay over a distance of the order of its initial streamwise length. As $t \to \infty$ the left-hand side of (13) will hence tend towards $\overline{u}$, and the inequality may then be written, in this limit, as

$$L\int_{-\infty}^{\infty} u^2\,dx \leqslant \overline{u}^2, \tag{15}$$

where

$$L = ([U]+2\Delta)\,t \tag{16}$$

and

$$[U] = U_{\max} - U_{\min}.$$

Thus for the total integrated kinetic energy of the disturbance one finds in the limit of $t \to \infty$,

$$\frac{1}{2}\int_{-\infty}^{\infty} (u^2+v^2+w^2)\,dx > \tfrac{1}{2}tU'^2\overline{v}_0^2/([U]+2\Delta) \tag{17}$$

for any $\Delta > 0$, i.e., the total kinetic energy of a disturbance with $\hat{v}_0 \neq 0$ in an inviscid parallel shear flow will grow at least as fast as linearly in time as $t \to \infty$.

3. Discussion

The result (17) holds irrespectively of whether the shear flow is stable or not to exponential growth of wavelike disturbances. Thus, any inviscid parallel shear flow will exhibit growth of the kinetic energy of a three-dimensional disturbance with $\overline{v}_0 \neq 0$. For a flow without an inflexion point this growth of kinetic energy is associated with the behaviour of the u component. An examination of this component for large times (see appendix) reveals that the u component remains bounded for large times, for any initial disturbance which is of finite extent in x, but the streamwise dimension of the disturbed region grows linearly in time. Three-dimensional disturbance with $\overline{v}_0 \neq 0$ will therefore evolve into long streaks of streamwise momentum excess or defect. It follows from (12) that this tendency could be especially strong where U' is large. That the simple linearized inviscid theory indicates this behaviour may be of significance in connexion with the longitudinal streaks, commonly observed in the wall region of turbulent and transitional boundary layers.

It should be re-emphasized that, in contrast to the case of a disturbance which is independent of x, such as was considered by Stuart (1965) and by Ellingsen & Palm (1975), an initial disturbance of finite streamwise extent need not lead to an eventual complete redistribution of the streamwise momentum because the total cross-stream displacement of individual fluid elements remains finite for a weak disturbance in this case, according to linear theory. One may estimate this displacement from the result (12) taking the u perturbation to be associated primarily with a y displacement, Δy, of the fluid element, and the length of the disturbed region to be of order $[U]t$. Thereby one finds (provided $U' \neq 0$)

$$\Delta y = O(\overline{v}_0/[U]).$$

For linearization to be valid one must require this to be small compared to the width d of the shear flow, or

$$\bar{v}_0 \ll d[U].$$

The right-hand side of this inequality is a measure of the streamwise momentum of the shear flow available for redistribution; hence the linearized theory does not hold if the net vertical momentum of the initial disturbance is comparable in magnitude with the streamwise momentum. Therefore, linear theory is very much restricted for the analysis of x-independent disturbances, for which the vertical momentum is infinite. Other nonlinear effects, such as nonlinear distortion and secondary instability due to formation of thin shear layers, may also limit the validity of the linearized theory. A simple kinematical argument shows that such nonlinear effects could become large at times for which an individual fluid element has been convected by the horizontal perturbation velocity flow field a distance comparable with the horizontal dimensions of the region of initial disturbance. Thus, linearized theory would only be valid for

$$q_0' t \ll l,$$

where q_0' is the initial horizontal velocity perturbation and l a typical horizontal dimension of the initially perturbed region. Hence the linear growth of perturbation energy analysed here may hold only during a finite time interval inversely proportional to the magnitude of the initial perturbation. Such general restrictions in the validity of linearized theory for the analysis of initial disturbances in a shear flow must of course always be kept in mind when interpreting the results. Also, viscous effects will become increasingly important for large times. An analysis based on a purely convected disturbance (with $v = 0$) shows that the inviscid analysis is limited to times short compared to $(l^2/\nu U'^2)^{\frac{1}{3}}$ (Landahl 1977).

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Appendix. Behaviour of a localized inviscid disturbance for large times

Let a caret denote a triple Fourier transform with respect to x, z and t, i.e.,

$$\hat{v}(y) = \iiint_{-\infty}^{\infty} e^{-i(\alpha x+\beta z-\omega t)}\, v(x,z,y,t)\, dx\, dz\, dt. \tag{A 1}$$

(Convergence of the time integral for the causal initial-value problem may require that ω must have a positive imaginary part.) Application of Fourier transform to (6), (1) and (5) gives, respectively,

$$(U-c)(\hat{v}''-k^2\hat{v}) - U''\hat{v} = \tilde{f}_0/i\alpha, \tag{A 2}$$

$$\hat{u} = -\frac{\hat{v}U' + i\alpha\hat{p}/\rho}{i\alpha(U-c)} + \frac{\tilde{u}_0}{i\alpha(U-c)}, \tag{A 3}$$

$$\hat{p}/\rho = -\frac{i\alpha}{k^2}[(U-c)\,\hat{v}' - U'\hat{v}] + \frac{\tilde{v}_0}{k^2}, \tag{A 4}$$

where

$$c = \omega/\alpha, \tag{A 5}$$

$$k^2 = \alpha^2+\beta^2, \tag{A 6}$$

$$\tilde{f}_0 = (\tilde{v}_0''-k^2\tilde{v}_0), \tag{A 7}$$

and tildes denote x, z transforms of the initial conditions, i.e.

$$\tilde{v}_0 = \iint_{-\infty}^{\infty} v(x,y,z,0)\, e^{-i(\alpha x+\beta z)}\, dx\, dz. \tag{A 8}$$

Combination of (A 3) and (A 4) gives

$$\hat{u} = -\frac{\beta^2\hat{v}U'}{i\alpha k^2(U-c)}+\frac{i\alpha}{k^2}\hat{v}'+\frac{\tilde{u}_0}{i\alpha(U-c)}-\frac{\tilde{v}_0'}{k^2(U-c)}. \tag{A 9}$$

We shall investigate the behaviour of u for large times. It will be assumed that the initial velocity field is localized in space and is smooth with continuous first derivatives. The last two terms in (A 9) involving the initial velocity field are easily inverted yielding a contribution which may be written as a function of y, z and the convected co-ordinate $x-Ut$. This function, if finite at $t = 0$ for all x, will remain finite for all t. The first term is of the form $\hat{f} = \hat{g}/[i\alpha(U-c)]$, the inverse of which is given by

$$f = \int_0^t g[x-U(t-t_1), y, z, t_1]\, dt_1, \tag{A 10}$$

where f and g are the inverse transforms of $\hat{f}$ and $\hat{g}$, respectively. Application of this to (A 9) shows that boundedness of v for $t \to \infty$, which has been the main concern in traditional hydrodynamic stability theory, does not necessarily guarantee boundedness of u; this depends on the behaviour of v for large values of both $|x|$ and t. Furthermore, u may remain non-zero in the limit $t\to\infty$ even when $v \to 0$, so that it becomes necessary to consider also velocity profiles which are stable in the usual hydrodynamic sense, i.e., profiles without inflexion. For unstable flows with inflexion, v will grow exponentially with time, and the estimate (17) will then be highly conservative and rather uninteresting in this context. Therefore, the case without inflexional instability will be given special attention here.

In studying the behaviour of v for large $|x|$ and t we may make use of techniques familiar in earlier works on hydrodynamic stability.

A formal solution for $\hat{v}$ may be constructed from two linearly independent homogeneous solutions, $\hat{v} = \Phi_{1,2}$, of (A 2) in the usual fashion by the method of variation of parameters (or equivalently, from one-dimensional Green's functions)

$$\hat{v} = \int_0^y [\Phi_1(y)\,\Phi_2(y_1)-\Phi_2(y)\,\Phi_1(y_1)]\,[i\alpha(U_1-c)\,W]^{-1}\tilde{f}_0(y_1)\,dy_1+\hat{C}_1\,\Phi_1(y)+\hat{C}_2\,\Phi_2(y), \tag{A 11}$$

where W is the Wronskian, $W = \Phi_1'\,\Phi_2-\Phi_2'\,\Phi_1$ and $U_1 \equiv U(y_1)$. The integration constant $\hat{C}_1$ and $\hat{C}_2$ are to be chosen such that the boundary conditions

$$\hat{v} = 0 \quad \text{at} \quad y = 0, d, \tag{A 12}$$

are satisfied. Two linearly independent solutions Φ_1 and Φ_2 may be constructed by use of Heisenberg's (1924) technique in form of the series (see Lin 1955, p. 34)

$$\Phi_{1,2} = (U-c) \sum_{n=0}^{\infty} q_n^{(1,2)}(y)\, k^{2n}, \tag{A 13}$$

where

$$q_{n+1}^{(1,2)} = \int_0^y (U_1-c)^{-2}\, dy_1 \int_{y_c}^{y_1} (U_2-c)^2 q_n^{(1,2)}\, dy_2, \tag{A 14}$$

$$q_0^{(1)} = 1, \tag{A 15}$$

$$q_0^{(2)} = \int_0^y (U_1-c)^{-2}\, dy_1, \tag{A 16}$$

and where $U_1 \equiv U(y_1)$, $U_2 \equiv U(y_2)$, $U(y_c) = c$. The solution Φ_1 is regular in c. The main singularities are contained in the first term (A 16) in the series for Φ_2. By expansion around the double pole at $y = y_c$ one finds, assuming $U(0) = 0$,

$$q_2^{(2)} \cong -\frac{U}{U_c' c(U-c)} - \frac{U-c}{U_c'^3} \ln\left(\frac{U-c}{-c}\right) + 0(c \ln c) + 0[(U-c)\ln(U-c)]. \tag{A 17}$$

Substitution into (A 14) shows that all other singularities are of higher order. In order for the solution to represent properly an initial-value problem one has to assume that αc has a small positive imaginary part. The proper branch of the logarithmic term in (A 17) is then to be selected by letting the integration path in the complex y plane pass below the singular point for $U'(y_c) > 0$ and above it for $U'(y_c) < 0$.

The Wronskian is a constant for the Rayleigh equation. It is most easily evaluated at $y = y_c$, and one finds

$$W = -F_{1c}, \tag{A 18}$$

where index c refers to the value at $y = y_c$,

$$F_{1c} = 1 + k^2 \int_0^{y_c} (U_1-c)^{-2}\, dy_1 \int_{y_c}^{y_1} (U_1-c)^2\, dy_2 + \dots.$$

Substitution into (A 11) and application of the boundary condition at $y = d$ yields a solution which may be written in the form

$$\hat{v} = \int_0^d \hat{G}(y, y_1) \tilde{f}_0(y_1)\, dy_1, \tag{A 19}$$

where

$$\hat{G}(y,y_1) = \begin{cases} \Phi_2(y_1)\,[\Phi_1(y) - \Phi_1(d)\,\Phi_2(y)/\Phi_2(d)]\,[i\alpha(U_1-c)\,W]^{-1} & \text{for} \quad y < y_1, \\ \Phi_2(y)\,[\Phi_1(y_1) - \Phi_1(d)\,\Phi_2(y_1)/\Phi_2(d)]\,[i\alpha(U_1-c)\,W]^{-1} & \text{for} \quad y > y_1. \end{cases} \tag{A 20}$$

$\hat{G}$ is seen to have a pole at $c = U_1$ and branch points *at* $c = U_1$ and $c = U(y)$. Additional poles may arise from the zeroes of $\Phi_2(d)$, which correspond to the eigenvalues in the usual stability problem for an infinite wave train. The asymptotic behaviour for large time and x is determined by these poles in the c and α planes, the branch points only contributing with terms varying as $t^{-n} (n \geqslant 1)$ for $t \to \infty$. Inversion of the contribution from the pole at $c = U_1$ yields, in consistency with (A 10),

$$\mathscr{F}^{-1}\{[i\alpha(U_1-c)]^{-1}\} = \delta(x - U_1 t)\, H(t), \tag{A 21}$$

where δ is the Dirac delta function and $H(t)$ the unit step function. Substitution into (A 19) then gives, upon integration over y_1, a contribution to v vanishing as t (provided $U_1' \neq 0$) in a convected frame of reference travelling with a velocity within the range of $U(y)$. For fixed x, v will vanish faster, as t^{-2}. Also, for fixed t, v will vanish for $|x| \to \infty$. Inversion of (A 9) with the aid of (A 10) then shows that this gives a contribution to u which remains finite as $t \to \infty$.

Let the poles corresponding to $\Phi_2(d) = 0$ be located at

$$c = c^{(n)}(k) = c_{\mathscr{R}}^{(n)} + ic_{\mathscr{I}}^{(n)}. \tag{A 22}$$

From (A 16), (A 17) is follows that $c_{\mathscr{R}}^{(n)}$ must lie within the range of $U(y)$ for small $|c_{\mathscr{I}}^{(n)}|$, otherwise $\Phi_{2\mathscr{R}}$ becomes of one sign. Inversion with respect to time of this pole contribution will give a term of the form

$$e^{-i\alpha c^{(n)}t}. \tag{A 23}$$

Of interest here is the case of stable waves, i.e., flows without any inflexion points, so that $\alpha c_{\mathscr{I}}^{(n)} < 0$. Such eigenvalues can arise for the initial-value problem when the proper branch of the integral (A 16) is taken† (see statement below (A 17)). At the inversion with respect to x, the contribution from around $\alpha = 0$ yields

$$\frac{1}{\pi} \frac{-c_{\mathscr{I}}^{(n)}t}{(x - c_{\mathscr{R}}^{(n)}t)^2 + (c_{\mathscr{I}}^{(n)}t)^2}, \tag{A 24}$$

where $c_{\mathscr{I}}^{(n)}$ and $c_{\mathscr{R}}^{(n)}$ are to be evaluated at $\alpha = 0 (k = |\beta|)$. For small $c_{\mathscr{I}}^{(n)}/c_{\mathscr{R}}^{(n)}$ and moderate times this behaves like a delta function of the argument $x - c_{\mathscr{R}}^{(n)}t$, and for large t it gives a contribution vanishing as t^{-1} in either a fixed or a convected frame of reference.

That v is found to tend to zero for large t as t^{-1} is consistent with $\bar{v}$ being independent of time and the perturbed region growing linearly in time. The asymptotic behaviour of $\bar{v}$ for large $|x|$ and t thus found assures that u remains bounded as $t \to \infty$ for a flow without inflexion. For a three-dimensional disturbance with $\bar{v}_0 \neq 0$, u will remain non-zero in the limit of $t \to \infty$, but in the two-dimensional case, for which $\bar{v} = 0$ by continuity, one finds that v will vanish as t^{-2} rather than as t^{-1}, and u will decay in the same manner. Application of continuity also shows that, in as much as the u field will elongate in the streamwise direction in proportion to t so that $\partial u/\partial x \sim t^{-1}$, w will hence also tend towards zero as t^{-1}, except possibly in the streamwise tails of the disturbance region, whose dimensions are set by the streamwise scale of the initial distribution.

As a simple example, consider the case of constant shear, $U = U'y$, for which one may take $\Phi_1 = \exp(-ky)$, $\Phi_2 = \exp(ky)$. For this case there is no normal mode, and after inversion the exact solution for v is found to be given by

$$v = -\frac{1}{4\pi} \int\int\int_{-\infty}^{\infty} \frac{f_0(x_1, y_1, z_1)\, dx_1\, dy_1\, dz_1}{[(\xi - x_1)^2 + (y - y_1)^2 + (z - z_1)^2]^{\frac{1}{2}}}, \tag{A 25}$$

† The proof of the Rayleigh criterion, which shows that there are no eigenvalues with $c_{\mathscr{I}} < 0$ for an eigenfunction with continuous first derivative, does not exclude the possibility of poles $c = c^{(n)}$, $\Phi_2(d) = 0$ in the lower complex plane for the initial-value problem. For this problem the inversion integral is to be carried out along a contour slightly above the real c axis (for $\alpha > 0$). Along this path $\hat{v}$ has a branch point for $c = U(y)$, and in order to specify $\hat{v}$ uniquely for all c, a branch cut must be introduced in the complex c plane, most conveniently from $c = U(y)$ to $c = U(y) - i\infty$. Therefore, for $c_{\mathscr{I}} < 0$ and $c_{\mathscr{R}}$ within the range of U, $\hat{v}$ and $\hat{v}'(y)$ may be discontinuous for some $y = y_c$ with $U(y_c) = c_{\mathscr{R}}$.

where

$$\xi = x - U'y_1 t.$$

Expansion for large t yields

$$v \sim \frac{1}{2\pi U' t}\int_{-\infty}^{\infty} g_0(y_c, z_1)\ln[(y-y_c)^2+(z-z_1)^2]^{\frac{1}{2}}\,dz_1, \tag{A 26}$$

where

$$g_0(y,z) = \int_{-\infty}^{\infty} f_0(x,y,z)\,dx \tag{A 27}$$

and

$$y_c = x/U't. \tag{A 28}$$

The cross-flow for large times thus becomes a two-dimensional one in each cross-flow plane x = const. decaying as t^{-1}. For $g_0 = 0$, as would be the case for a two-dimensional (in x, y plane) flow, v will decay as t^{-2} or faster.

By use of (A 9) the solution for u may be constructed and expanded for large t in a similar manner. One finds, after some calculations,

$$u \sim \frac{1}{2\pi}\int_{x/U't}^{\pm\infty} \frac{g_0(y_1,z_1)}{(y-y_1)}\ln[(y-y_1)^2+(z-z_1)^2]^{\frac{1}{2}}\,dy_1\,dz_1, \tag{A 29}$$

where the upper sign should be used for $y < x/U't$ and the lower one for $y > x/U't$. This approximation does not hold for $y \simeq x/U't$, for which a thin shear layer develops with a structure depending on the detailed f_0 distribution. It follows from (A 29) that the front of the disturbance propagates with a velocity equal to the maximum one, U_{max}, for the region of non-zero f_0. Similarly, the back of the disturbance will propagate with the minimum velocity, U_{min}, in the y range of non-zero f_0 so that the disturbed region will grow in the streamwise direction at a rate given by $U_{max} - U_{min}$.

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